

# Kepler Problem in the Constant Curvature Space.

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## Abstract

We present algebraic derivation of the result of Schrödinger [1] for the spectrum of hydrogen atom in the space with constant curvature.

## 1 Introduction

The theory of different mechanical systems, both classical and quantum in the space of constant curvature attracts attention since long. Especially interesting to study the systems like Hydrogen atom, initiated by Schrödinger [1] in this curved space, as it may have cosmological application. A collection of references concerning these issues could be found in [2]. Here we are not going to discuss neither motivation, no application of this system. What we are going to consider is the dynamical symmetry of Hydrogen atom in the space with constant curvature in classical and especially in quantum case and, as it is possible to do in the flat case [3] to derive the spectrum of the system by pure algebraic tools. It happened to be quite unexpected that in spite of similarity of two system — in flat and curved spaces, the construction of its generators in the curved case is not at all straightforward.

The 3-dimensional space with constant curvature are sphere  $S^3$ , hyperboloids  $H^3, H'^3$  and cone  $V^3$ . For definiteness we shall consider the case of positive curvature e.d.  $S^3$ . The coordinates of  $S^3$  will be  $X_\alpha$ ,  $\alpha = 1, 2, 3, 4$ ,  $X_\alpha^2 = 1$ . The simplest mapping of  $R^3$  ( $R^3/\{\infty\}$ ) into  $S^3$  is stereographic projection given by

$$X_\alpha = \left( \frac{2\lambda \mathbf{x}}{\mathbf{x}^2 + \lambda^2}, \frac{\mathbf{x}^2 - \lambda^2}{\mathbf{x}^2 + \lambda^2} \right) \quad (1.1)$$

The only possible kinetic term, invariant with respect to group of motion of  $S^3$ — $SO(4)$  arises if we take the Lagrangian proportional to the square of angular momentum tensor  $M_{\alpha\beta} = X_\alpha \dot{X}_\beta - \dot{X}_\alpha X_\beta$ :

$$L = \frac{m}{2} \frac{1}{4\lambda^2} M_{\alpha\beta}^2 = \frac{m}{2} \frac{\dot{\mathbf{x}}^2}{(\mathbf{x}^2 + \lambda^2)^2} \quad (1.2)$$

The kinetic term of the Hamiltonian, which corresponds to this Lagrangian is given by

$$H = \frac{1}{2m} \mathbf{p}^2 (\mathbf{x}^2 + \lambda^2)^2 \quad (1.3)$$

This Hamiltonian has 6 integrals of motion, which are generators of its group of motion  $SO(4)$ :

$$\begin{aligned} L_i &= \epsilon_{ijk} x_j p_k, \\ K_i &= \frac{1}{2\lambda} (p_i (\mathbf{x}^2 - \lambda^2) - 2x_i \mathbf{p} \mathbf{x}), \end{aligned}$$

The canonical Poisson brackets for  $p_i, x_i$  induce the algebra  $SO(4)$  for generators  $L_i, K_i$ :

$$\begin{aligned} \{L_i, L_j\} &= -\epsilon_{ijk} L_k \\ \{K_i, K_j\} &= -\epsilon_{ijk} L_k \\ \{L_i, K_j\} &= -\epsilon_{ijk} K_k \end{aligned}$$

The Hamiltonian  $H$  proportional to one Casimir:

$$H = \frac{1}{2m} 4\lambda^2 (\mathbf{L}^2 + \mathbf{K}^2), \quad (1.4)$$

while the second  $\mathbf{L} \mathbf{K} = \mathbf{0}$ .

Now we shall consider Hamiltonian with Kepler potential [1],[2]

$$H = \frac{1}{2m} \mathbf{p}^2 (\mathbf{x}^2 + \lambda^2)^2 + \frac{2\lambda\alpha}{m} \frac{\mathbf{x}^2 - \lambda^2}{|\mathbf{x}|} \quad (1.5)$$

As in flat case, the Hamiltonian (1.5) possesses 6 integrals of motion (of course they are not all independent): the angular momentum  $\mathbf{L}$  and Laplace-Runge-Lenz (LRL) vector  $\mathbf{A}$

$$\mathbf{A} = \mathbf{K} \times \mathbf{L} + \alpha \frac{\mathbf{x}}{|\mathbf{x}|}, \quad (1.6)$$

apparently  $\mathbf{A}$  and  $\mathbf{L}$  satisfy the relation

$$\mathbf{A} \mathbf{L} = \mathbf{0} \quad (1.7)$$

The algebra of Poisson brackets of vectors  $\mathbf{A}, \mathbf{L}$  has the following form:

$$\{A_i, A_j\} = \epsilon_{ijk} L_k \left( \frac{m}{2\lambda^2} H - 2\mathbf{L}^2 \right), \quad \{L_i, A_j\} = -\epsilon_{ijk} A_k. \quad (1.8)$$

The square of vector  $\mathbf{A}$  is expressed via  $\mathbf{L}^2$  and Hamiltonian:

$$\mathbf{A}^2 = \alpha^2 - \mathbf{L}^4 + \frac{m}{2\lambda^2} \mathbf{L}^2 H. \quad (1.9)$$

The reader, acquainted with properties of LRL in the flat case [3] immediately sees the difference both in algebra and the relation of  $\mathbf{A}^2$  with  $H$  and  $\mathbf{L}^2$ . This difference does not allow us to define in a simple way the vector which forms together with  $\mathbf{L}$  the  $SO(3) \times SO(3)$  algebra which is dynamical symmetry responsible for supplementary degeneration in quantum mechanics. But this construction is however possible and consideration of its classical analog shows the way towards it in quantum case. First of all let us introduce the rescaled Hamiltonian  $h$ :

$$h = \frac{m}{2\lambda^2} H. \quad (1.10)$$

From equation (1.9) we obtain

$$\mathbf{A}^2 = -[\mathbf{L}^2 - (\frac{h}{2} + \sqrt{\frac{h^2}{4} + \alpha^2})][\mathbf{L}^2 - (\frac{h}{2} - \sqrt{\frac{h^2}{4} + \alpha^2})] \quad (1.11)$$

and, because  $\mathbf{A}^2 > 0$ ,  $\mathbf{L}^2$  satisfies the following inequality:

$$0 \leq \mathbf{L}^2 \leq (\frac{h}{2} + \sqrt{\frac{h^2}{4} + \alpha^2}). \quad (1.12)$$

Now let us find the Poisson brackets of  $\mathbf{A}f(\mathbf{L}^2)$  (the function  $f(\mathbf{L}^2)$  could be  $h$ -dependent)

$$\{A_i f(\mathbf{L}^2), A_j f(\mathbf{L}^2)\} = \epsilon_{ijk} L_k \frac{\partial(\mathbf{A}^2 f^2(\mathbf{L}^2))}{\partial \mathbf{L}^2} \quad (1.13)$$

In order for (1.13) be the part of  $SO(3) \times SO(3)$  algebra the following condition should be satisfied

$$\mathbf{A}^2 f^2(\mathbf{L}^2) = a - \mathbf{L}^2. \quad (1.14)$$

Apparently, if we take  $a = (\frac{h}{2} + \sqrt{\frac{h^2}{4} + \alpha^2})$  the R.H.S of (1.14) will be positive for any point in the phase space because of inequality (1.12). Moreover, the function  $f(\mathbf{L}^2)$  for this choice of  $a$  will be essentially simplified

$$f(\mathbf{L}^2) = \sqrt{\frac{a - \mathbf{L}^2}{\mathbf{A}^2}} = \left[ \mathbf{L}^2 - (\frac{h}{2} - \sqrt{\frac{h^2}{4} + \alpha^2}) \right]^{-\frac{1}{2}} \quad (1.15)$$

Thus, the vector

$$\mathbf{R} = \mathbf{A} \left[ \mathbf{L}^2 - \left( \frac{h}{2} - \sqrt{\frac{h^2}{4} + \alpha^2} \right) \right]^{-\frac{1}{2}} \quad (1.16)$$

together with angular momentum  $\mathbf{L}$  forms the desired algebra  $SO(3) \times SO(3)$ :

$$\begin{aligned} \{L_i, L_j\} &= -\epsilon_{ijk} L_k, \\ \{R_i, R_j\} &= -\epsilon_{ijk} L_k, \\ \{L_i, R_j\} &= -\epsilon_{ijk} R_k. \end{aligned} \quad (1.17)$$

Further, the following expression for nontrivial Casimir of  $SO(3) \times SO(3)$  holds true:

$$\mathbf{R}^2 + \mathbf{L}^2 = \left( \frac{h}{2} + \sqrt{\frac{h^2}{4} + \alpha^2} \right), \quad (1.18)$$

the other Casimir  $\mathbf{R}\mathbf{L}$  vanishes. From (1.18) we obtain

$$h = \mathbf{R}^2 + \mathbf{L}^2 - \frac{\alpha^2}{\mathbf{R}^2 + \mathbf{L}^2}. \quad (1.19)$$

This is the result we were looking for and which is of a great importance in quantum case. The obvious obstacle for quantum generalization of this construction is non-commutativity of operators  $\mathbf{A}$  and  $\mathbf{L}^2$ . This will be the problem we are going to solve below.

## 2 Quantum Theory

The Hilbert space of quantum theory of Kepler problem in the space of constant curvature  $S^3$  is the space  $L^2$  on  $S^3$ , where the scalar product  $\langle \phi | \psi \rangle$  is given by

$$\langle \phi | \psi \rangle = \int d^4 X \bar{\phi}(X) \psi(X) \delta(X^2 - 1), \quad (2.1)$$

where  $X$  are coordinates in 4D space  $X_\alpha, \alpha = 1, 2, 3, 4$ . Using the map  $R^3$  into  $S^3$  given by (1.1) we can express the scalar product (2.1) up to inessential factor as an integral over  $R^3$ :

$$\langle \phi | \psi \rangle = \int d^3 x \frac{1}{(\mathbf{x}^2 + \lambda^2)^3} \bar{\phi}(\mathbf{x}) \psi(\mathbf{x}) \quad (2.2)$$

The measure in this scalar product makes non-trivial definition of operators we need for Kepler problem. Apart from making these operators Hermitian we have to preserve their algebraic properties. As an example let us consider the vector  $\mathbf{K}$ . If we define the quantum operator as

$$K_i = \frac{1}{2\lambda} [(\mathbf{x}^2 - \lambda^2)p_i - 2x_i \mathbf{x} \mathbf{p}], \quad (2.3)$$

where the operator  $\mathbf{p} = -i\frac{\partial}{\partial \mathbf{x}}$  (note, that  $\mathbf{p}$  is not Hermitian), then  $\mathbf{K}$  will be Hermitian and its commutation relations will be the same, as in classical case:

$$[K_i, K_j] = i\epsilon_{ijk} L_k \quad (2.4)$$

The operator  $\mathbf{L}$  brings no difficulty because it commutes with measure in (2.2). Having both operators  $\mathbf{K}, \mathbf{L}$  properly defined we can find the kinetic part of quantum Hamiltonian:

$$\mathbf{K}^2 + \mathbf{L}^2 = -\frac{(\mathbf{x}^2 + \lambda^2)^3}{4\lambda^2} \frac{\partial}{\partial x_i} \frac{1}{(\mathbf{x}^2 + \lambda^2)} \frac{\partial}{\partial x_i}, \quad (2.5)$$

which is apparently Hermitian. Thus, the total Hamiltonian has the following form:

$$H = \frac{2\lambda^2}{m} \left[ \mathbf{K}^2 + \mathbf{L}^2 + \frac{\alpha}{\lambda} \frac{\mathbf{x}^2 - \lambda^2}{|\mathbf{x}|} \right] = \frac{2\lambda^2}{m} h \quad (2.6)$$

Now we are ready to define the quantum LRL vector

$$\mathbf{A} = \frac{1}{2} (\mathbf{K} \times \mathbf{L} - \mathbf{L} \times \mathbf{K}) + \alpha \frac{\mathbf{x}}{|\mathbf{x}|} \quad (2.7)$$

With this ordering the operator  $\mathbf{A}$  commutes with Hamiltonian:

$$[h, \mathbf{A}] = \mathbf{0} \quad (2.8)$$

The commutator of different components of  $\mathbf{A}$  are the same as in classical case:

$$[A_i, A_j] = -i\epsilon_{ijk} L_k (h - 2\mathbf{L}^2), \quad (2.9)$$

while its square is slightly different

$$\mathbf{A}^2 = \alpha^2 + h(\mathbf{L}^2 + 1) - (\mathbf{L}^2 + 1)^2 + 1. \quad (2.10)$$

Now we will be looking for the operator  $\mathbf{R}$ , such that

$$[R_i, R_j] = i\epsilon_{ijk}L_k \quad (2.11)$$

in the form, which does not destroy the Hermicity of  $\mathbf{R}$ :

$$R_i = f^{1/2}(\mathbf{L}^2)A_i f^{1/2}(\mathbf{L}^2) \quad (2.12)$$

Further it will be convenient to use instead of operator  $\mathbf{L}^2$  its function  $\gamma$ :

$$\gamma = \sqrt{(\mathbf{L}^2 + \frac{1}{4})} - \frac{1}{2}, \quad \mathbf{L}^2 = \gamma(\gamma + 1). \quad (2.13)$$

Also we shall need formula, which is proven in the **Appendix**, valid for any vector operator  $\mathbf{A}$ , such that  $\mathbf{A}\mathbf{L} = \mathbf{0}$ :

$$\begin{aligned} A_i f(\gamma) &= \frac{f(\gamma + 1)}{2\gamma + 1} [(\gamma + 1)A_i + i\epsilon_{ijk}L_j A_k] \\ &\quad + \frac{f(\gamma - 1)}{2\gamma + 1} [\gamma A_i - i\epsilon_{ijk}L_j A_k] \end{aligned} \quad (2.14)$$

Now we are ready to calculate the commutator (2.11):

$$\begin{aligned} &[f^{1/2}(\gamma)A_i f^{1/2}(\gamma), f^{1/2}(\gamma)A_j f^{1/2}(\gamma)] \\ &= f^{1/2}(\gamma) (A_i f(\gamma)A_j - A_j f(\gamma)A_i) f^{1/2}(\gamma). \end{aligned} \quad (2.15)$$

To calculate the parenthesis in the R.H.S. of (2.15) we use the equation (2.14) and after some algebraic transformation we arrive at

$$\begin{aligned} &(A_i f(\gamma)A_j - A_j f(\gamma)A_i) = \\ &i\epsilon_{ijk}L_k \left( \frac{f(\gamma+1)}{2\gamma+1}(\gamma r(\gamma) - \mathbf{A}^2) + \frac{f(\gamma-1)}{2\gamma+1}((\gamma+1)r(\gamma) + \mathbf{A}^2) \right), \end{aligned} \quad (2.16)$$

where we have introduces function  $r(\gamma)$  through

$$[A_i, A_j] = i\epsilon_{ijk}L_k r(\gamma) \quad (2.17)$$

Making use of this result we proceed with calculation of commutator (2.11) and move all  $f^{1/2}(\gamma)$  to the right, because they commute with  $\mathbf{L}$

$$[R_i, R_j] = i\epsilon_{ijk}L_k T, \quad (2.18)$$

where  $T$  is given by

$$T = \frac{f(\gamma)f(\gamma+1)}{2\gamma+1}(\gamma r(\gamma) - \mathbf{A}^2) + \frac{f(\gamma)f(\gamma-1)}{2\gamma+1}((\gamma+1)r(\gamma) + \mathbf{A}^2) \quad (2.19)$$

Our goal is to make  $T = 1$ . Having expressions for  $r(\gamma)$  and  $\mathbf{A}^2$

$$\begin{aligned} r(\gamma) &= 2\gamma(\gamma+1) - h \\ \mathbf{A}^2 &= \alpha^2 + h(\gamma(\gamma+1) + 1) - \gamma^2(\gamma+1)^2 - 2\gamma(\gamma+1), \end{aligned} \quad (2.20)$$

we can rewrite the equation  $T = 1$  in the following form:

$$\begin{aligned} f(\gamma)f(\gamma+1)[\alpha^2 + h(\gamma+1)^2 - (\gamma+1)^2((\gamma+1)^2 - 1)] \\ - f(\gamma)f(\gamma-1)[\alpha^2 + h\gamma^2 - \gamma^2(\gamma^2 - 1)] = -(2\gamma+1). \end{aligned} \quad (2.21)$$

The first and the second terms in the L.H.S. of (2.21) differ by shift of  $\gamma$  by 1, therefore we immediately obtain

$$f(\gamma)f(\gamma-1) = \frac{\mu - \gamma^2}{\alpha^2 + h\gamma^2 - \gamma^2(\gamma^2 - 1)}, \quad (2.22)$$

where  $\mu$  does not depend of  $\gamma$ . Before solving this equation for  $f(\gamma)$  let us find the square of operator  $\mathbf{R}$

$$\mathbf{R}^2 = f^{1/2}(\gamma)A_i f(\gamma)A_i f^{1/2}(\gamma). \quad (2.23)$$

Using again equation (2.14) we can move  $A_i$  to the right through  $f(\gamma)$  and after some algebraic transformation we arrive at

$$\begin{aligned} \mathbf{R}^2 &= \frac{f(\gamma)f(\gamma+1)}{2\gamma+1}(\gamma+1)[\mathbf{A}^2 - \gamma(\gamma)] \\ &+ \frac{f(\gamma)f(\gamma-1)}{2\gamma+1}\gamma[\mathbf{A}^2 + (\gamma+1)r(\gamma)] \end{aligned} \quad (2.24)$$

Here we can use the result (2.22) and complete this calculation

$$\mathbf{R}^2 = \mu - 1 - \gamma(\gamma+1), \quad (2.25)$$

therefore we get

$$\mathbf{R}^2 + \mathbf{L}^2 = \mu - 1 \quad (2.26)$$

the result which was expected. Now it is time to choose  $\mu$ . The equation (2.22) could be written as follows

$$\begin{aligned} f(\gamma)f(\gamma-1) &= \frac{\mu - \gamma^2}{(k_1 - \gamma^2)(\gamma^2 - k_2)}, \\ k_{1,2} &= \frac{h+1}{2} \pm \sqrt{\frac{(h+1)^2}{4} + \alpha^2}. \end{aligned} \quad (2.27)$$

This formula wakes the reminiscence of the classical case. The choice

$$\mu = \frac{h+1}{2} + \sqrt{\frac{(h+1)^2}{4} + \alpha^2} \quad (2.28)$$

gives us

$$\mathbf{R}^2 + \mathbf{L}^2 = -1 + \frac{h+1}{2} + \sqrt{\frac{(h+1)^2}{4} + \alpha^2} \quad (2.29)$$

and as a result, the expression for Hamiltonian via  $\mathbf{R}^2 + \mathbf{L}^2$ :

$$h = \mathbf{R}^2 + \mathbf{L}^2 - \frac{\alpha^2}{\mathbf{R}^2 + \mathbf{L}^2 + 1}. \quad (2.30)$$

Now let us discuss this representation of Hamiltonian. The operators  $\mathbf{R}$  and  $\mathbf{L}$  forms the algebra  $SO(4)$ , which is the direct sum of two algebras  $SO(3)$ , so we can introduce instead of  $\mathbf{R}$  and  $\mathbf{L}$  another pair of operators  $\mathbf{M}, \mathbf{N}$

$$\mathbf{M} = \frac{1}{2}(\mathbf{L} + \mathbf{R}), \quad \mathbf{N} = \frac{1}{2}(\mathbf{L} - \mathbf{R}) \quad (2.31)$$

such that

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k, & [N_i, N_j] &= i\epsilon_{ijk}N_k \\ [M_i, N_j] &= 0. \end{aligned} \quad (2.32)$$

The two Casimirs of  $S0(4)$  in terms of  $\mathbf{M}, \mathbf{N}$  are

$$C_1 = \mathbf{M}^2, \quad C_2 = \mathbf{N}^2 \quad (2.33)$$

and because  $\mathbf{RL} = \mathbf{0}$ ,

$$C_1 = C_2 = \frac{1}{4}(\mathbf{R}^2 + \mathbf{L}^2). \quad (2.34)$$



The spectrum of  $C_1(C_2)$  is given by  $k(k+1)$ ,  $k = 0, \frac{1}{2}, 1, \dots$ . The representation, characterized by  $k$  contains all angular momenta  $l = 0, 1, \dots, 2k$  with multiplicity 1. As a result, the spectrum of Hamiltonian  $h$  will be given by

$$h = 4k(k+1) - \frac{\alpha^2}{4k(k+1)+1} = 2k(2k+2) - \frac{\alpha^2}{(2k+1)^2}. \quad (2.35)$$

Introducing another quantum number  $n = 2k+1$ ,  $n = 1, 2, \dots$ , we can present (2.35) in the form

$$h = (n-1)(n+1) - \frac{\alpha^2}{n^2} \quad (2.36)$$

which coincides with Schödinger's result [1].

The last thing which has to be mentioned here is the explicit form of function  $f(\gamma)$ , although we can easily avoid its construction. Using equation (2.14) we can find vector  $\mathbf{R}$ , knowing only bilinear combinations like  $f(\gamma)f(\gamma \pm 1)$ . But for completeness we shall produce the result especially because it is worth to be mentioned. With our choice of  $\mu$  — (2.28), the equation (2.22) takes the following form:

$$f(\gamma)f(\gamma-1) = \frac{1}{(\gamma^2 + \sqrt{\frac{(h+1)^2}{4} + \alpha^2} - \frac{h+1}{2})}. \quad (2.37)$$

The expression  $\sqrt{\frac{(h+1)^2}{4} + \alpha^2} - \frac{h+1}{2}$  is always positive, so denoting it as  $\rho^2$ , we have

$$f(\gamma)f(\gamma-1) = \frac{1}{\gamma^2 + \rho^2} \quad (2.38)$$

The solution of this equation is given by

$$f(x) = \frac{i}{x-i\rho} \frac{\Gamma(\frac{x-i\rho+1}{2})}{\Gamma(\frac{x-i\rho}{2})} \frac{\Gamma(-\frac{x+i\rho}{2})}{\Gamma(-\frac{x+i\rho-1}{2})}. \quad (2.39)$$

Proof could be done by direct substitution into equation.

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## A Appendix

Let us consider a function  $F(\mathbf{L}^2)$  and multiply it from the left by an operator  $\mathbf{A}$  which is a vector with respect to  $\mathbf{L}$  and satisfies the condition  $\mathbf{A}\mathbf{L} = \mathbf{0}$ . In general the following relation exists:

$$A_i F(\mathbf{L}^2) = S(\mathbf{L}^2) A_i + T(\mathbf{L}^2) i\epsilon_{ijk} L_j A_k \quad (\text{A.1})$$

where the functions  $S, T$  is defined by  $F$ . The limitation to the case of  $\mathbf{A}\mathbf{L} = \mathbf{0}$  is not essential, but as we do not need this general case for our purpose. Let the function  $F(\mathbf{L}^2)$  be represented in the form

$$F(\mathbf{L}^2) = \int d\alpha \phi(\alpha) e^{i\alpha \mathbf{L}^2} \quad (\text{A.2})$$

Thus, in order to derive (A.1) we need to consider only the case of exponent of  $\mathbf{L}^2$ . Multiplying the exponent by  $\mathbf{A}$  from the left, we have

$$A_i e^{i\alpha \mathbf{L}^2} = f(\alpha, \mathbf{L}^2) A_i + g(\alpha, \mathbf{L}^2) i\epsilon_{ijk} L_j A_k, \quad (\text{A.3})$$

where the functions  $f, g$  have to be defined. Now let us differentiate both sides of (A.3) over  $\alpha$

$$\begin{aligned} \partial_\alpha A_i e^{i\alpha \mathbf{L}^2} &= i A_i e^{i\alpha \mathbf{L}^2} \mathbf{L}^2 = i [f(\alpha, \mathbf{L}^2) A_i + g(\alpha, \mathbf{L}^2) i\epsilon_{ijk} L_j A_k] \mathbf{L}^2 \\ &= i [f(\alpha, \mathbf{L}^2)(\mathbf{L}^2 + 2) + 2g(\alpha, \mathbf{L}^2)\mathbf{L}^2] A_i \\ &\quad + i [2f(\alpha, \mathbf{L}^2) + g(\alpha, \mathbf{L}^2)] i\epsilon_{ijk} L_j A_k \\ &= \partial_\alpha f(\alpha, \mathbf{L}^2) A_i + \partial_\alpha g(\alpha, \mathbf{L}^2) i\epsilon_{ijk} L_j A_k, \end{aligned} \quad (\text{A.4})$$

where we have used the equation

$$A_i \mathbf{L}^2 = (\mathbf{L}^2 + 2) A_i + 2i\epsilon_{ijk} L_j A_k \quad (\text{A.5})$$

From (A.4) follow two equation for functions  $f, g$ :

$$\begin{aligned}\partial_\alpha f(\alpha, \mathbf{L}^2) &= i [f(\alpha, \mathbf{L}^2)(\mathbf{L}^2 + 2) + 2g(\alpha, \mathbf{L}^2)\mathbf{L}^2] \\ \partial_\alpha g(\alpha, \mathbf{L}^2) &= i [2f(\alpha, \mathbf{L}^2) + g(\alpha, \mathbf{L}^2)], \\ f(0, \mathbf{L}^2) &= 1, \quad g(0, \mathbf{L}^2) = 0\end{aligned}\tag{A.6}$$

Further it will be convenient to introduce instead of  $\mathbf{L}^2$  the operator  $\gamma$

$$\gamma = \sqrt{(\mathbf{L}^2 + \frac{1}{4})} - \frac{1}{2}, \quad \mathbf{L}^2 = \gamma(\gamma + 1).\tag{A.7}$$

In terms of  $\gamma$  the solution of (A.6) has the following form:

$$\begin{aligned}f(\alpha, \gamma(\gamma + 1)) &= \frac{1}{2\gamma + 1} [(\gamma + 1)e^{i\alpha(\gamma+1)(\gamma+2)} + \gamma e^{i\alpha\gamma(\gamma-1)}] \\ g(\alpha, \gamma(\gamma + 1)) &= \frac{1}{2\gamma + 1} [e^{i\alpha(\gamma+1)(\gamma+2)} - e^{i\alpha\gamma(\gamma-1)}].\end{aligned}\tag{A.8}$$

Substituting (A.8) into (A.3) we receive:

$$\begin{aligned}A_i e^{i\alpha\gamma(\gamma+1)} &= e^{i\alpha(\gamma+1)(\gamma+2)} \frac{1}{2\gamma + 1} [(\gamma + 1)A_i + i\epsilon_{ijk}L_j A_k] \\ &\quad + e^{i\alpha\gamma(\gamma-1)} \frac{1}{2\gamma + 1} [\gamma A_i - i\epsilon_{ijk}L_j A_k].\end{aligned}\tag{A.9}$$

Thus, moving the vector operator through the exponent of  $\gamma(\gamma + 1)$  produces two term, one with  $\gamma$  shifted by  $+1$ , the other by  $-1$ . Would we had the operator  $\mathbf{A}$  such that  $\mathbf{A}\mathbf{L} \neq \mathbf{0}$  the third term will appear, where  $\gamma$  will not be shifted. Making the Fourier transformation we extend the result for arbitrary function

$$\begin{aligned}A_i F(\gamma) &= F(\gamma + 1) \frac{1}{2\gamma + 1} [(\gamma + 1)A_i + i\epsilon_{ijk}L_j A_k] \\ &\quad + F(\gamma - 1) \frac{1}{2\gamma + 1} [\gamma A_i - i\epsilon_{ijk}L_j A_k].\end{aligned}\tag{A.10}$$

The analogous formula exists also for right multiplication.

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